

An example of unitary equivalence between self-adjoint extensions and their parameters

KONSTANTIN PANKRASHKIN

Laboratoire de mathématiques – UMR 8628

Université Paris-Sud 11, Bâtiment 425

91400 Orsay, France

Homepage: <http://www.math.u-psud.fr/~pankrash/>

E-mail: konstantin.pankrashkin@math.u-psud.fr

Abstract

The spectral problem for self-adjoint extensions is studied using the machinery of boundary triplets. For a class of symmetric operators having Weyl functions of a special type we calculate explicitly the spectral projections in the form of operator-valued integrals. This allows one to give a constructive proof of the fact that, in certain intervals, the resulting self-adjoint extensions are unitarily equivalent to the parameterizing boundary operator acting in a smaller space, and one is able to provide an explicit form for the associated unitary transform. Applications to differential operators on metric graphs and to direct sums are discussed.

Keywords: self-adjoint extension, unitary equivalence, metric graph, Laplacian, boundary conditions, boundary triplet, Weyl function

MSC 2010 classification: 47A56, 47B25, 34B45.

1 Introduction

Let S be a closed densely defined symmetric operator in a separable Hilbert space \mathcal{H} with equal deficiency indices $n_{\pm}(S) = n$; it is a well known fact that the self-adjoint extensions of S are then parameterized by the unitary operators or self-adjoint linear relations in the n -dimensional Hilbert space [26]. From the point of view of quantum mechanics, the choice of a particular self-adjoint extension can be viewed as a quantization problem, as different extensions correspond to Hamiltonians defining different quantum dynamics [25], and understanding the role of the extension parameters in the properties of the associated Hamiltonians is one of the central problems. Questions of the control of the point spectrum in terms of parameters were addressed already by Krein [30] for the case of finite deficiency indices, and it was then extended in a series of papers to more sophisticated situations including the study of singular and absolutely continuous spectra, see the discussion in [3, 10]. The progress in the spectral analysis of self-adjoint extensions was mainly possible due to the concept of boundary triplet and associated Weyl functions [17, 18, 26], whose main points we briefly recall now. A *boundary triplet* [26] for S consists of an auxiliary Hilbert space \mathcal{G} and two linear maps $\Gamma, \Gamma' : \text{dom } S \rightarrow \mathcal{G}$ satisfying the following two conditions:

- $\langle f, S^*g \rangle_{\mathcal{H}} - \langle S^*f, g \rangle_{\mathcal{H}} = \langle \Gamma f, \Gamma'g \rangle_{\mathcal{G}} - \langle \Gamma'f, \Gamma g \rangle_{\mathcal{G}}$ for all $f, g \in \text{dom } S^*$,
- the application $(\Gamma, \Gamma') : \text{dom } S^* \ni f \mapsto (\Gamma f, \Gamma'f) \in \mathcal{G} \oplus \mathcal{G}$ is surjective.

The boundary triplets provide an efficient way of dealing with the self-adjoint extensions of the operator S . We restrict ourselves by considering two distinguished self-adjoint extensions, namely,

$$H^0 := S^* \upharpoonright \ker \Gamma, \quad H := S^* \upharpoonright \ker \Gamma';$$

here and below $A \upharpoonright L$ denotes the restriction of A to L . In our setting H^0 will be considered as a reference operator, and H is viewed as its perturbation, and we will show below that various situations are reduced to the above case, and the parameters may be included into the boundary triplet.

An essential role in the analysis of the self-adjoint extensions is played by the so-called Weyl function $M(z)$ which is defined as follows [17]. It is known that for $z \notin \text{spec } H^0$ the operator

$$\gamma(z) := (\Gamma \upharpoonright \ker(S^* - z))^{-1}$$

is well defined, and it is a linear topological isomorphism between \mathcal{G} and $\ker(S^* - z) \subset \mathcal{H}$. The map

$$\mathbb{C} \setminus \text{spec } H^0 \ni z \mapsto \gamma(z) \in \mathcal{L}(\mathcal{G}, \mathcal{H}),$$

which is usually called the γ -field, is holomorph. The holomorph operator function

$$\mathbb{C} \setminus \text{spec } H^0 \ni z \mapsto M(z) := \Gamma' \gamma(z) \in \mathcal{L}(\mathcal{G})$$

is referred to as the Weyl function associated with the boundary triplet. The following result is well known [17]:

Theorem 1. *For any $z \notin \text{spec } H^0$ there holds*

$$\ker(H - z) = \gamma(z) \ker M(z). \quad (1)$$

Moreover, for $z \notin \text{spec } H^0 \cup \text{spec } H$ the Krein resolvent formula holds,

$$(H - z)^{-1} = (H^0 - z)^{-1} - \gamma(z)M(z)^{-1}\gamma(\bar{z})^*. \quad (2)$$

A direct consequence of the resolvent formula (2) is the relation

$$\text{spec } H \setminus \text{spec } H^0 = \left\{ z \notin \text{spec } H^0 : 0 \in \text{spec } M(z) \right\}. \quad (3)$$

Numerous papers were dedicated to the study of possible refinements of (3) in order to recover the spectral nature of H in terms of the Weyl function, see e.g. the discussion in [2, 10, 11, 17, 18, 31]. Such an analysis becomes deeper if some additional properties of H^0 and M are available. In the present paper we assume that the following two conditions hold:

- the operator H^0 has a spectral gap, i.e. there exists an open interval $J := (a_0, b_0) \subset \mathbb{R} \setminus \text{spec } H^0$, and

- the Weyl function M can be represented as

$$M(z) = \frac{m(z) - T}{n(z)}, \quad (4)$$

where T is a bounded self-adjoint operator in \mathcal{G} and m, n are scalar functions which are holomorph outside $\text{spec } H^0$, with n non-vanishing in J .

Such situations arise in several important applications such as the analysis of differential operators on metric graphs, this will be reviewed in Section 4. In this case, Eq. (3) gives a rather simple characterization of the spectrum of H in J :

$$\text{spec } H \cap J = \{\lambda \in J : m(\lambda) \in \text{spec } T\}.$$

This relation was refined in [13] in order to describe all spectral types of H :

$$\text{spec}_\star H \cap J = \{\lambda \in J : m(\lambda) \in \text{spec}_\star T\}, \quad \star \in \{p, pp, \text{disc}, \text{ess}, \text{ac}, \text{sc}\}. \quad (5)$$

A further improvement was obtained first in [2] for the particular case $n = \text{const}$ and then extended in [37] to general denominators n , and we need to introduce an additional notation to formulate it. Let $\mathcal{B}(\mathbb{R})$ denote the set of the borelian subsets of \mathbb{R} . For a self-adjoint operator A acting in a Hilbert space \mathcal{H} we denote by $E_A : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ the operator-valued spectral measure associated with A , i.e. for any $\Omega \in \mathcal{B}(\mathbb{R})$ we have $E_A(\Omega) := 1_\Omega(A)$, where $1_\Omega : \mathbb{R} \rightarrow \mathbb{R}$ is the indicator function of Ω . For the same Ω , we denote by A_Ω the operator $AE_A(\Omega)$ viewed as a self-adjoint operator in the Hilbert space $\text{ran } E_A(\Omega)$ equipped with the induced scalar product. Using this notation, the main result of [2, 37] can be formulated as follows: If the Weyl function admits the representation (4), then $m : J \cap \text{spec } H \rightarrow m(J \cap \text{spec } H)$ is a bijection, and the operator H_J is unitarily equivalent to $m^{-1}(T_{m(J)})$. This indeed implies the spectral relations (5).

One has to emphasize that the approach suggested in [2] and then used in [37] was rather implicit. In particular, the unitary equivalence of the two operators was shown using the so-called generalized Naimark theorem on minimal orthogonal dilations of operator-valued measures [33], and no information on the unitary transform was obtained. Finding this transform in an explicit form is the subject of the present work. The main results of the present paper can be summarized as follows (see Theorem 10 and Corollary 11):

- for any borelian subset $\Omega \subset J$ one has the equality $E_H(\Omega) = UE_T(m(\Omega))U^*$, where U is the operator given by

$$U = \int_J \sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda) dE_T(m(\lambda))$$

understood as an improper operator-valued Riemann-Stieltjes integral (the precise meaning will be discussed in detail in Sections 2 and 3).

- Moreover, U viewed as a map from $\text{ran } E_T(m(J))$ to $\text{ran } E_H(J)$ is unitary, and $m(H_J) = UT_{m(J)}U^*$.

As we will see below, in various situations the operator T plays the role of a parameter of self-adjoint extensions, so the above results provide a direct translation of the spectral properties of the parameters to the spectral properties of the associated self-adjoint extensions.

We believe that the knowledge of a certain explicit form for the unitary transform U relating the operators H and T provides a considerable progress compared to the previously known results. In particular, it allows one to reduce the functional calculus for H to that for T and, for example, to make a link between various evolution problems associated with the two operators. It should be emphasized that the approach used in the present paper differs from the one employed in the previous works [2, 37], we calculate the spectral projections for H in a rather direct way using the limit values of the resolvent, which allows one to write an explicit formula for the unitary transform in question. While the situation we look at is indeed very special, we show below that it admits some useful applications in mathematical physics, see Section 4. Moreover, we are not aware of any previous work giving any explicit expression for the spectral projections of suitably large classes of self-adjoint extensions in a closed form, and we believe that the approach presented here may admit some generalizations to more sophisticated Weyl functions and might shed a new light on the spectral analysis and the functional calculus of self-adjoint extensions.

2 Basic formula for spectral projections

The aim of the present section is to express the spectral projections for H in terms of the spectral measure for T . The main formula is given in Lemma 4, and it will be used for the subsequent spectral analysis. Throughout the rest of the paper we use the notation

$$S_T := [\inf \operatorname{spec} T, \sup \operatorname{spec} T] \subset \mathbb{R}.$$

The following simple properties of m and n were proved in [37, Section 2.2]:

Lemma 2. *Denote $K := m^{-1}(S_T) \cap J$, then*

- $n(x) \cdot m'(x) > 0$ for all $x \in K$,
- K is a connected set.

The maps $\gamma(z)$ and $M(z)$ satisfy a number of identities, see e.g. [13, 17]. In particular,

$$M(z)^* = M(\bar{z}), \quad z \notin \operatorname{spec} H^0, \quad (6)$$

$$M(z_1) - M(\bar{z}_2) = (z_1 - \bar{z}_2)\gamma(z_2)^*\gamma(z_1), \quad z_1, z_2 \notin \operatorname{spec} H^0, \quad (7)$$

$$\Im M(z) > 0 \text{ for } \Im z > 0.$$

The last property means that M is a strict operator-valued Nevanlinna-Herglotz function. Recall that for any operator-valued Nevanlinna-Herglotz function F in \mathcal{G} and any $f \in \mathcal{G}$, the map

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\} \ni z \mapsto F_f(z) := \langle f, F(z)f \rangle_{\mathcal{G}}$$

is a scalar Nevanlinna-Herglotz function (called also R -function), hence for almost all $x \in \mathbb{R}$ there exists a finite limit $\lim_{\varepsilon \rightarrow 0+} \langle f, F(x + i\varepsilon)f \rangle_{\mathcal{G}}$. Using the polar identity we see that, for any $f, g \in \mathcal{G}$, the limit $\lim_{\varepsilon \rightarrow 0+} \langle f, F(x + i\varepsilon)g \rangle_{\mathcal{G}}$ is finite for almost all $x \in \mathbb{R}$.

The following proposition will not be used in the rest of the text, but it provides us with a certain intuition in the study of the spectral projections for H .

Proposition 3. For any $\lambda \in J \cap \text{spec}_p H$ there holds

$$E_H(\{\lambda\}) = \frac{n(\lambda)}{m'(\lambda)} \gamma(\lambda) E_T(\{m(\lambda)\}) \gamma(\lambda)^*.$$

Proof. Let (e_j) be an orthonormal basis in $\ker(T - m(\lambda))$. It follows from (1) that the family $\left(\sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda) e_j\right)$ is a basis in $\ker(H - \lambda)$, and we just need to show that this basis is orthonormal. To see this, take any $\xi_1, \xi_2 \in \ker(T - m(\lambda))$, then, using (7),

$$\begin{aligned} \left\langle \sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda) \xi_1, \sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda) \xi_2 \right\rangle_{\mathcal{H}} &= \frac{n(\lambda)}{m'(\lambda)} \langle \xi_1, \gamma(\lambda)^* \gamma(\lambda) \xi_2 \rangle_{\mathcal{G}} \\ &= \frac{n(\lambda)}{m'(\lambda)} \langle \xi_1, M'(\lambda) \xi_2 \rangle_{\mathcal{G}} = \frac{n(\lambda)}{m'(\lambda)} \left\langle \xi_1, \left(\frac{m'(\lambda)}{n(\lambda)} - \frac{n'(\lambda)}{n(\lambda)^2} \cdot (m(\lambda) - T) \right) \xi_2 \right\rangle_{\mathcal{G}} \\ &= \frac{n(\lambda)}{m'(\lambda)} \left\langle \xi_1, \frac{m'(\lambda)}{n(\lambda)} \xi_2 \right\rangle_{\mathcal{G}} = \langle \xi_1, \xi_2 \rangle_{\mathcal{G}}. \end{aligned}$$

which concludes the proof. \square

The aim of the present section is to find a certain analog of Proposition 3 for operators with continuous spectra. Take an arbitrary segment $[a, b] \subset J$ whose endpoints satisfy

$$a, b \notin \text{spec}_p H \text{ or, equivalently, } m(a), m(b) \notin \text{spec}_p T, \quad (8)$$

and consider the operator

$$\Phi([a, b]) := \int_a^b \sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda) dE_T(m(\lambda)) \in \mathcal{L}(\mathcal{G}, \mathcal{H}) \quad (9)$$

and its adjoint

$$\Phi^* \equiv \Phi([a, b])^* := \int_a^b \sqrt{\frac{n(\lambda)}{m'(\lambda)}} dE_T(m(\lambda)) \gamma(\lambda)^* \in \mathcal{L}(\mathcal{H}, \mathcal{G}).$$

These two operators are defined as suitable limits of the associated Riemann-Stieltjes integral sums. More precisely, consider a partition Δ of $[a, b]$, $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$, denote $\Delta_j := [\lambda_{j-1}, \lambda_j]$, pick some $\xi_j \in \overline{\Delta_j}$ and consider the Riemann-Stieltjes integral sum

$$\Phi_{\Delta} = \sum_{j=1}^n \sqrt{\frac{n(\xi_j)}{m'(\xi_j)}} \gamma(\xi_j) E_T(m(\Delta_j)). \quad (10)$$

The strong limit of Φ_{Δ} as $\max_j |\Delta_j| \rightarrow 0$, if it exists and is independent of the choice of the partition points λ_j , ξ_j , is then denoted by the above integral expression (9). The general theory of such integrals is rather involved, see e.g. [5, 9], but in our case the maps γ are holomorph, and the study of the above integral reduces to the study of usual (scalar) Riemann-Stieltjes integral with respect to spectral measures. The existence in our case can be proved in an easier way using the constructions of [1, Section 7]. Note that the

points $m(\lambda)$ with $m'(\lambda) = 0$ do not belong to the support of E_T due to Lemma 2 and that the map $\lambda \mapsto \sqrt{\frac{n(\lambda)}{m'(\lambda)}} \gamma(\lambda)$ is lipschitzian on $[a, b] \cap m^{-1}(S_T)$. This is sufficient to show that, for $\max_j |\Delta_j| \rightarrow 0$, the above integral sums Φ_Δ and their adjoints Φ_Δ^* converge in the operator norm to their limit values $\Phi \equiv \Phi([a, b])$ and $\Phi([a, b])^*$ respectively, see [1, Remark 7.3]. The following lemma is the main result of the section and provides the main technical ingredient of the subsequent discussion.

Lemma 4. *Let a segment $[a, b] \subset J$ satisfy (8), then $E_H([a, b]) = \Phi([a, b])\Phi([a, b])^*$.*

Proof. Throughout the proof we denote for brevity

$$P := E_H([a, b]), \quad \Phi := \Phi([a, b]) \quad \text{and} \quad \Pi := \Phi\Phi^*.$$

So we need to prove the equality $P = \Pi$. By the Stone formula for the spectral projections, cf. [27, Theorem 42], we have

$$P = \text{s-lim}_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \Im(H - x - i\varepsilon)^{-1} dx,$$

Let us assume first that

$$m'(x) \neq 0 \text{ for all } x \in [a, b], \quad (11)$$

As the map $z \mapsto (H^0 - z)^{-1}$ is holomorph in a complex neighborhood of $[a, b]$, one has

$$\text{s-lim}_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \Im(H^0 - x - i\varepsilon)^{-1} dx = 0.$$

Using the resolvent formula (2) we arrive at

$$P = -\text{s-lim}_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \Im \left[\gamma(x + i\varepsilon) M(x + i\varepsilon)^{-1} \gamma(x - i\varepsilon)^* \right] dx. \quad (12)$$

Let \mathcal{D} be the algebraic linear hull of the set $\{\gamma(z)\xi : z \in \mathbb{C} \setminus \mathbb{R}, \xi \in \mathcal{G}\}$ and let $f \in \mathcal{D}^\perp$. First, for all $z \notin \mathbb{R}$ there holds $\gamma(z)^* f = 0$, and Eq. (12) gives $Pf = 0$. On the other hand, the map $z \mapsto \gamma(z)^* f$ extends by continuity to $z \in [a, b]$, which shows that $\gamma(\lambda)^* f = 0$ for all $\lambda \in [a, b]$, which, in its turn, gives the equalities $\Phi^* f = 0$ and $\Pi f = 0$. Therefore, the operators P and Π coincide on \mathcal{D}^\perp .

Now let us show that P coincides with Π on the closure of \mathcal{D} . As both P and Π are bounded operators, it is now sufficient to prove that $\langle f, Pg \rangle_{\mathcal{H}} = \langle f, \Pi g \rangle_{\mathcal{H}}$ for the vectors f and g having the form $f = \gamma(z_1)\xi_1$ and $g = \gamma(z_2)\xi_2$ with arbitrary $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$ and $\xi_1, \xi_2 \in \mathcal{G}$. For such f and g we have

$$\begin{aligned} \langle f, Pg \rangle_{\mathcal{H}} &= -\lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_a^b \left\langle \gamma(z_1)\xi_1, \Im \left[\gamma(x + i\varepsilon) M(x + i\varepsilon)^{-1} \gamma(x - i\varepsilon)^* \right] \gamma(z_2)\xi_2 \right\rangle_{\mathcal{H}} dx \\ &= -\frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \int_a^b \left\langle \xi_1, \gamma(z_1)^* \Im \left[\gamma(x + i\varepsilon) M(x + i\varepsilon)^{-1} \gamma(x - i\varepsilon)^* \right] \gamma(z_2)\xi_2 \right\rangle_{\mathcal{G}} dx. \end{aligned}$$

Using (6) and (7) we have

$$\begin{aligned}
2i\gamma(z_1)^*\Im\left[\gamma(x+i\varepsilon)M(x+i\varepsilon)^{-1}\gamma(x-i\varepsilon)^*\right]\gamma(z_2) \\
&= \gamma(z_1)^*\gamma(x+i\varepsilon)M(x+i\varepsilon)^{-1}\gamma(x-i\varepsilon)^*\gamma(z_2) \\
&\quad - \gamma(z_1)^*\gamma(x-i\varepsilon)M(x-i\varepsilon)^{-1}\gamma(x+i\varepsilon)^*\gamma(z_2) \\
&= \frac{M(x+i\varepsilon)-M(\bar{z}_1)}{x+i\varepsilon-\bar{z}_1}M(x+i\varepsilon)^{-1}\frac{M(z_2)-M(x+i\varepsilon)}{z_2-x-i\varepsilon} \\
&\quad - \frac{M(x-i\varepsilon)-M(\bar{z}_1)}{x-i\varepsilon-\bar{z}_1}M(x-i\varepsilon)^{-1}\frac{M(z_2)-M(x-i\varepsilon)}{z_2-x+i\varepsilon} \\
&= \frac{M(z_2)+M(\bar{z}_1)-M(x+i\varepsilon)-M(\bar{z}_1)M(z_2)M(x+i\varepsilon)^{-1}}{(x+i\varepsilon-\bar{z}_1)(z_2-x-i\varepsilon)} \\
&\quad - \frac{M(z_2)+M(\bar{z}_1)-M(x-i\varepsilon)-M(\bar{z}_1)M(z_2)M(x-i\varepsilon)^{-1}}{(x-i\varepsilon-\bar{z}_1)(z_2-x+i\varepsilon)}.
\end{aligned}$$

As the map $z \mapsto M(z)$ is holomorph in a certain complex neighborhood of $[a, b]$ and z_1 and z_2 are fixed non-real numbers, we have the obvious relations

$$\begin{aligned}
\lim_{\varepsilon \rightarrow +0} \int_a^b \frac{M(z_2)+M(\bar{z}_1)-M(x+i\varepsilon)}{(x+i\varepsilon-\bar{z}_1)(z_2-x-i\varepsilon)} dx &= \lim_{\varepsilon \rightarrow +0} \int_a^b \frac{M(z_2)+M(\bar{z}_1)-M(x-i\varepsilon)}{(x-i\varepsilon-\bar{z}_1)(z_2-x+i\varepsilon)} dx \\
&= \int_a^b \frac{M(z_2)+M(\bar{z}_1)-M(x)}{(x-\bar{z}_1)(z_2-x)} dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle f, Pg \rangle_{\mathcal{H}} &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow +0} \int_a^b \left\langle \xi_1, \left[\frac{M(\bar{z}_1)M(z_2)M(x-i\varepsilon)^{-1}}{(\bar{z}_1-x+i\varepsilon)(z_2-x+i\varepsilon)} \right. \right. \\
&\quad \left. \left. - \frac{M(\bar{z}_1)M(z_2)M(x+i\varepsilon)^{-1}}{(\bar{z}_1-x-i\varepsilon)(z_2-x-i\varepsilon)} \right] \xi_2 \right\rangle_{\mathcal{G}} dx.
\end{aligned}$$

Furthermore, we have the representation

$$\frac{1}{(\bar{z}_1-x-i\varepsilon)(z_2-x-i\varepsilon)} = \frac{1+\varepsilon h(x, \varepsilon)}{(\bar{z}_1-x)(z_2-x)},$$

where h is a continuous function with $h(x, 0) = 1$ and such that for some $\varepsilon_0 > 0$ one has the bound $h_0 := \sup_{x \in [a, b], |\varepsilon| < \varepsilon_0} |h(x, \varepsilon)| < \infty$. We have then

$$\begin{aligned}
\langle f, Pg \rangle_{\mathcal{G}} &= \lim_{\varepsilon \rightarrow +0} \int_a^b \left\langle \xi_1, \frac{M(\bar{z}_1)M(z_2)}{(\bar{z}_1-x)(z_2-x)} \cdot \frac{M(x-i\varepsilon)^{-1}-M(x+i\varepsilon)^{-1}}{2\pi i} \xi_2 \right\rangle_{\mathcal{G}} dx \\
&\quad - \lim_{\varepsilon \rightarrow +0} \int_a^b \frac{\varepsilon h(x, -\varepsilon) \left\langle M(z_1)\xi_1, M(x-i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}}}{2\pi i(\bar{z}_1-x)(z_2-x)} dx \\
&\quad - \lim_{\varepsilon \rightarrow +0} \int_a^b \frac{\varepsilon h(x, \varepsilon) \left\langle M(z_1)\xi_1, M(x+i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}}}{2\pi i(\bar{z}_1-x)(z_2-x)} dx.
\end{aligned} \tag{13}$$

By elementary considerations, see [13, Lemma 3.14], for small but non-zero ε we have the representation

$$M(x + i\varepsilon)^{-1} = n(x + i\varepsilon)L(x, \varepsilon)(m(x) + i\varepsilon m'(x) - T)^{-1},$$

where $L(x, \varepsilon) \in \mathcal{L}(\mathcal{G})$ such that $\lim_{\varepsilon \rightarrow 0} \sup_{x \in [a, b]} \|L(x, \varepsilon) - 1\|_{\mathcal{L}(\mathcal{G})} = 0$. By the assumption (11) made at the beginning, we have the bound

$$\kappa := \inf_{x \in [a, b]} |m'(x)| > 0.$$

and for sufficiently small $\varepsilon \neq 0$ and all $x \in [a, b]$ one has the uniform estimate, with some fixed $C > 0$,

$$\left| \frac{\varepsilon h(x, \pm\varepsilon)}{(\bar{z}_1 - x)(z_2 - x)} \left\langle M(z_1)\xi_1, M(x \pm i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}} \right| \leq C.$$

On the other hand, as noted at the beginning of the section, for almost all $x \in \mathbb{R}$ there exist finite limits

$$\lim_{\varepsilon \rightarrow 0+} \left\langle M(z_1)\xi_1, M(x \pm i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}} =: C_{\pm}(x).$$

Therefore, by using the dominated convergence we arrive at

$$\begin{aligned} & \lim_{\varepsilon \rightarrow +0} \int_a^b \frac{\varepsilon h(x, \pm\varepsilon)}{(\bar{z}_1 - x)(z_2 - x)} \left\langle M(z_1)\xi_1, M(x \pm i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}} dx \\ &= \int_a^b \lim_{\varepsilon \rightarrow +0} \frac{\varepsilon h(x, \pm\varepsilon)}{(\bar{z}_1 - x)(z_2 - x)} \left\langle M(z_1)\xi_1, M(x \pm i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}} dx \\ &= \int_a^b \lim_{\varepsilon \rightarrow +0} \frac{\varepsilon h(x, \pm\varepsilon)}{(\bar{z}_1 - x)(z_2 - x)} \lim_{\varepsilon \rightarrow +0} \left\langle M(z_1)\xi_1, M(x \pm i\varepsilon)^{-1}M(z_2)\xi_2 \right\rangle_{\mathcal{G}} dx \\ &= \int_a^b \lim_{\varepsilon \rightarrow +0} \frac{\varepsilon h(x, \pm\varepsilon)}{(\bar{z}_1 - x)(z_2 - x)} C_{\pm}(x) dx = 0. \end{aligned}$$

Substituting these equalities into (13) we arrive at

$$\langle f, Pg \rangle_{\mathcal{H}} = \lim_{\varepsilon \rightarrow +0} \int_a^b \left\langle \xi_1, \frac{M(\bar{z}_1)M(z_2)}{(\bar{z}_1 - x)(z_2 - x)} \cdot \frac{M(x - i\varepsilon)^{-1} - M(x + i\varepsilon)^{-1}}{2\pi i} \xi_2 \right\rangle_{\mathcal{G}} dx \quad (14)$$

Denote by μ the spectral measure for the operator T associated with the vectors $M(z_1)\xi_1$

and $M(z_2)\xi_2$. With the help of μ one can rewrite the equality (14) as follows:

$$\begin{aligned}
& \int_a^b \left\langle \xi_1, \frac{M(\bar{z}_1)M(z_2)}{(\bar{z}_1 - x)(z_2 - x)} \cdot \frac{M(x - i\varepsilon)^{-1} - M(x + i\varepsilon)^{-1}}{2\pi i} \xi_2 \right\rangle_{\mathcal{G}} dx \\
&= \int_a^b \frac{1}{(\bar{z}_1 - x)(z_2 - x)} \left\langle M(z_1)\xi_1, \frac{M(x - i\varepsilon)^{-1} - M(x + i\varepsilon)^{-1}}{2\pi i} M(z_2)\xi_2 \right\rangle_{\mathcal{G}} dx \\
&= \int_a^b \int_{S_T} \frac{1}{2\pi i(\bar{z}_1 - x)(z_2 - x)} \left(\frac{n(x + i\varepsilon)}{\lambda - m(x + i\varepsilon)} - \frac{n(x - i\varepsilon)}{\lambda - m(x - i\varepsilon)} \right) d\mu(\lambda) dx \quad (15) \\
&= \int_{S_T} k(\lambda, \varepsilon) d\mu(\lambda),
\end{aligned}$$

where we denoted

$$k(\lambda, \varepsilon) = \int_a^b \frac{1}{2\pi i(\bar{z}_1 - x)(z_2 - x)} \cdot \left(\frac{n(x + i\varepsilon)}{\lambda - m(x + i\varepsilon)} - \frac{n(x - i\varepsilon)}{\lambda - m(x - i\varepsilon)} \right) dx.$$

Our aim now is to pass to the limit $\varepsilon \rightarrow +0$ in the integral on the right-hand side of (15). By [37, Lemma 10] one has

$$\frac{1}{\lambda - m(x + i\varepsilon)} = \frac{1}{\lambda - m(x) - i\varepsilon m'(x)} \cdot \left(1 + \varepsilon g(x, \lambda, \varepsilon) \right),$$

with

$$\sup_{\substack{x \in [a, b], \lambda \in \mathbb{R} \\ 0 < |\varepsilon| < \varepsilon_0}} |g(x, \lambda, \varepsilon)| < +\infty.$$

As n is a holomorph function, one has the representation $n(x + i\varepsilon) = n(x) + \varepsilon p(x, \varepsilon)$ with

$$\sup_{\substack{x \in [a, b] \\ |\varepsilon| < \varepsilon_0}} |p(x, \varepsilon)| < +\infty.$$

Hence one can represent the above function k as

$$\begin{aligned}
k(\lambda, \varepsilon) &= \int_a^b \frac{n(x)}{2\pi i(\bar{z}_1 - x)(z_2 - x)} \cdot \left(\frac{1}{\lambda - m(x) - i\varepsilon m'(x)} - \frac{1}{\lambda - m(x) + i\varepsilon m'(x)} \right) dx \\
&+ \int_a^b \frac{1}{2\pi i(\bar{z}_1 - x)(z_2 - x)} \frac{\varepsilon r(x, \lambda, \varepsilon)}{\lambda - m(x) - i\varepsilon m'(x)} dx \\
&+ \int_a^b \frac{1}{2\pi i(\bar{z}_1 - x)(z_2 - x)} \frac{\varepsilon r(x, \lambda, -\varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)} dx \quad (16)
\end{aligned}$$

with $r(x, \lambda, \varepsilon) := p(x, \varepsilon)(1 + \varepsilon g(x, \lambda, \varepsilon)) + n(x)g(x, \lambda, \varepsilon)$, and one has the obvious bound

$$\sup_{\substack{x \in [a, b], \lambda \in \mathbb{R} \\ 0 < |\varepsilon| < \varepsilon_0}} |r(x, \lambda, \varepsilon)| =: R < +\infty.$$

Denote the three summands on the right-hand side of (16) by $I_1(\lambda, \varepsilon)$, $I_2(\lambda, \varepsilon)$ and $I_3(\lambda, \varepsilon)$, respectively. For all $\lambda \in \mathbb{R}$ and $0 < |\varepsilon| < \varepsilon_0$ we can estimate

$$\left| \frac{\varepsilon r(x, \lambda, \varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)} \right| \leq \frac{R}{\kappa},$$

implying the bound

$$|I_{2/3}(\lambda, \varepsilon)| \leq \frac{R|b-a|}{2\pi\kappa \cdot |\Im z_1 \Im z_2|} \text{ for all } \lambda \in \mathbb{R} \text{ and } 0 < |\varepsilon| < \varepsilon_0.$$

To study I_1 , we rewrite the above expression in the form

$$I_1(\lambda, \varepsilon) = \frac{1}{\pi} \int_a^b \frac{1}{(\bar{z}_1 - x)(z_2 - x)} \cdot \frac{\varepsilon m'(x)n(x)}{(\lambda - m(x))^2 + (\varepsilon m'(x))^2} dx.$$

Denoting $N := \sup_{x \in [a, b]} |n(x)|$ one obtains

$$\begin{aligned} |I_1| &\leq \frac{N}{\pi |\Im z_1 \Im z_2|} \int_a^b \frac{|m'(x)|}{(\lambda - m(x))^2 + \varepsilon^2 \kappa^2} dx = \frac{N}{\pi |\Im z_1 \Im z_2|} \left| \int_{m(a)}^{m(b)} \frac{\varepsilon}{(\lambda - y)^2 + \varepsilon^2 \kappa^2} dy \right| \\ &\leq \frac{N}{\pi |\Im z_1 \Im z_2|} \int_{-\infty}^{+\infty} \frac{\varepsilon}{y^2 + \varepsilon^2 \kappa^2} dy = \frac{N}{\kappa |\Im z_1 \Im z_2|}. \end{aligned}$$

Therefore, we have shown the estimate $\sup_{\lambda \in S_T, \varepsilon \in (0, \varepsilon_0)} |k(\lambda, \varepsilon)| < \infty$, and the dominated convergence gives

$$\langle f, Pg \rangle_{\mathcal{G}} = \lim_{\varepsilon \rightarrow 0+} \int_{S_T} k(\lambda, \varepsilon) d\mu(\lambda) = \int_{S_T} \lim_{\varepsilon \rightarrow 0+} k(\lambda, \varepsilon) d\mu(\lambda). \quad (17)$$

We are now going to calculate the limit $\lim_{\varepsilon \rightarrow 0+} k(\lambda, \varepsilon)$. Consider first the limits of the terms I_2 and I_3 . As noted above, the subintegral functions are uniformly bounded for small ε , and by applying the dominated convergence we obtain

$$\lim_{\varepsilon \rightarrow 0+} I_2(\lambda, \varepsilon) = \int_a^b \frac{1}{(\bar{z}_1 - x)(z_2 - x)} \lim_{\varepsilon \rightarrow 0+} \frac{\varepsilon r(x, \lambda, \varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)} dx.$$

For all but at most one $x \in [a, b]$ we have $\lambda \neq m(x)$ and

$$\lim_{\varepsilon \rightarrow 0+} \frac{\varepsilon r(x, \lambda, \varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)} = 0,$$

which gives $\lim_{\varepsilon \rightarrow 0+} I_2(\lambda, \varepsilon) = 0$. Similarly, $\lim_{\varepsilon \rightarrow 0+} I_3(\lambda, \varepsilon) = 0$

To calculate the limit of I_1 , without loss of generality we assume that $m'(x) > 0$ for all $x \in [a, b]$; otherwise one can change simultaneously the signs of T , m and n . Introduce

a new variable $y = m(x)$. By the implicit function theorem we have $x = \varphi(y)$ and $\varphi'(y) = 1/m'(x)$, and the expression for I_1 can be rewritten in the form

$$I_1(\lambda, \varepsilon) = \frac{1}{\pi} \int_{m(a)}^{m(b)} \frac{1}{(\bar{z}_1 - \varphi(y))(z_2 - \varphi(y))} \frac{\varepsilon n(\varphi(y))}{(\lambda - y)^2 + \frac{\varepsilon^2}{\varphi'(y)^2}} dy.$$

Introducing another new variable t by $y = \varepsilon t + \lambda$ we arrive at

$$I_1(\lambda, \varepsilon) = \frac{1}{\pi} \int_{\frac{m(a)-\lambda}{\varepsilon}}^{\frac{m(b)-\lambda}{\varepsilon}} \frac{1}{(\bar{z}_1 - \varphi(\varepsilon t + \lambda))(z_2 - \varphi(\varepsilon t + \lambda))} \frac{n(\varphi(\varepsilon t + \lambda))}{t^2 + \frac{1}{\varphi'(\varepsilon t + \lambda)^2}} dt.$$

One has the bounds

$$\sup_{\frac{m(a)-\lambda}{\varepsilon} \leq t \leq \frac{m(b)-\lambda}{\varepsilon}} |n(\varphi(\varepsilon t + \lambda))| = \sup_{a \leq x \leq b} |n(x)| = N < +\infty$$

and

$$\inf_{\frac{m(a)-\lambda}{\varepsilon} \leq t \leq \frac{m(b)-\lambda}{\varepsilon}} \frac{1}{\varphi'(\varepsilon t + \lambda)^2} = \inf_{a \leq x \leq b} m'(x)^2 = \kappa^2 > 0.$$

This leads to the estimate

$$\left| \frac{1}{(\bar{z}_1 - \varphi(\varepsilon t + \lambda))(z_2 - \varphi(\varepsilon t + \lambda))} \frac{n(\varphi(\varepsilon t + \lambda))}{t^2 + \frac{1}{\varphi'(\varepsilon t + \lambda)^2}} \right| \leq \frac{N}{|\Im z_1 \Im z_2|(t^2 + \kappa^2)}.$$

Using the fact that the function $t \mapsto (t^2 + \kappa^2)^{-1}$ belongs to $L^1(\mathbb{R})$ and applying the dominated convergence one obtains the equality

$$\lim_{\varepsilon \rightarrow 0+} I_1(\lambda, \varepsilon) = \frac{1}{\pi} \int_{\lim_{\varepsilon \rightarrow 0+} \frac{m(a)-\lambda}{\varepsilon}}^{\lim_{\varepsilon \rightarrow 0+} \frac{m(b)-\lambda}{\varepsilon}} \lim_{\varepsilon \rightarrow 0+} \frac{1}{(\bar{z}_1 - \varphi(\varepsilon t + \lambda))(z_2 - \varphi(\varepsilon t + \lambda))} \frac{n(\varphi(\varepsilon t + \lambda))}{t^2 + \frac{1}{\varphi'(\varepsilon t + \lambda)^2}} dt.$$

For any $a \neq 0$ there holds

$$\int_{-\infty}^0 \frac{dt}{a^2 + t^2} = \int_0^{+\infty} \frac{dt}{a^2 + t^2} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{2|a|},$$

and for any $c \in [a, b]$ one has

$$\lim_{\varepsilon \rightarrow 0+} \frac{m(c) - \lambda}{\varepsilon} = \begin{cases} +\infty, & \text{if } \lambda < m(c) \\ 0 & \text{if } \lambda = m(c) \\ -\infty, & \text{if } \lambda > m(c) \end{cases}.$$

Finally, note that for $\lambda \in [m(a), m(b)]$ there holds

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0+} \frac{1}{(\bar{z}_1 - \varphi(\varepsilon t + \lambda))(z_2 - \varphi(\varepsilon t + \lambda))} \frac{n(\varphi(\varepsilon t + \lambda))}{z^2 + \frac{1}{\varphi'(\varepsilon t + \lambda)^2}} \\ = \frac{1}{(\bar{z}_1 - \varphi(\lambda))(z_2 - \varphi(\lambda))} \frac{n(\varphi(\lambda))}{t^2 + \frac{1}{\varphi'(\lambda)^2}}. \end{aligned}$$

Putting all together we arrive at the equalities

$$\lim_{\varepsilon \rightarrow 0+} k(\lambda, \varepsilon) = \begin{cases} 0, & \text{if } \lambda \notin m([a, b]), \\ \frac{\varphi'(\lambda)n(\varphi(\lambda))}{2(\bar{z}_1 - \varphi(\lambda))(z_2 - \varphi(\lambda))}, & \text{if } \lambda \in \{m(a), m(b)\}, \\ \frac{\varphi'(\lambda)n(\varphi(\lambda))}{(\bar{z}_1 - \varphi(\lambda))(z_2 - \varphi(\lambda))}, & \text{if } \lambda \in m((a, b)). \end{cases}$$

Substituting these equalities into (17) and noting that, by assumption (8), we have $E_T(\{m(a)\}) = E_T(\{m(b)\}) = 0$ and $\mu(\{m(a)\}) = \mu(\{m(b)\}) = 0$, we arrive at

$$\begin{aligned} \langle f, Pg \rangle_{\mathcal{H}} &= \int_{m([a, b]) \cap S_T} \frac{\varphi'(\lambda)n(\varphi(\lambda))}{(\bar{z}_1 - \varphi(\lambda))(z_2 - \varphi(\lambda))} d\mu(\lambda) \\ &= \int_{[a, b] \cap m^{-1}(S_T)} \frac{1}{(\bar{z}_1 - \lambda)(z_2 - \lambda)} \frac{n(\lambda)}{m'(\lambda)} d\mu(m(\lambda)). \end{aligned}$$

Finally, using the inclusion $\text{supp } \mu \subset S_T$, we can rewrite it as

$$\langle f, Pg \rangle_{\mathcal{H}} = \left\langle M(z_1)\xi_1, \int_a^b \frac{1}{(\bar{z}_1 - \lambda)(z_2 - \lambda)} \frac{n(\lambda)}{m'(\lambda)} dE_T(m(\lambda)) M(z_2)\xi_2 \right\rangle_{\mathcal{G}}. \quad (18)$$

Now let us switch to the operator Π . Take a partition Δ of $[a, b]$, $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$, denote $\Delta_j := [\lambda_{j-1}, \lambda_j)$ and $\delta := \max |\Delta_j|$, and consider the integral sums Φ_Δ defined by (10) with $\xi_j := \lambda_j$. As Φ_Δ converge in the norm sense to Φ as δ tends to 0, the products $\Phi_\Delta \Phi_\Delta^*$ converge to Π . On the other hand, using the orthogonality property: $E(m(\Delta_j))E(m(\Delta_k)) = 0$ for $j \neq k$, we have the representation

$$\Phi_\Delta \Phi_\Delta^* = \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \gamma(\lambda_j) E_T(m(\Delta_j)) \gamma(\lambda_j)^*.$$

Therefore, with the help of the identity (7) we arrive at

$$\begin{aligned}
\langle f, \Pi g \rangle_{\mathcal{H}} &= \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \left\langle f, \gamma(\lambda_j) E_T(m(\Delta_j)) \gamma(\lambda_j)^* g \right\rangle_{\mathcal{G}} \\
&= \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \left\langle \gamma(\lambda_j)^* \gamma(z_1) \xi_1, E_T(m(\Delta_j)) \gamma(\lambda_j)^* \gamma(z_2) \xi_2 \right\rangle_{\mathcal{G}} \\
&= \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)(\bar{z}_1 - \lambda_j)(z_2 - \lambda_j)} \\
&\quad \cdot \left\langle (M(z_1) - M(\lambda_j)) \xi_1, E_T(m(\Delta_j)) (M(z_2) - M(\lambda_j)) \xi_2 \right\rangle_{\mathcal{G}}. \quad (19)
\end{aligned}$$

Denote

$$\begin{aligned}
n_0 &:= \min_{x \in [a, b]} |n(x)|, \quad n_1 := \max_{x \in [a, b]} |n(x)|, \\
m_0 &:= \min_{x \in [a, b]} |m'(x)|, \quad m_1 := \max_{x \in [a, b]} |m'(x)|, \quad A := \frac{n_1 m_1}{m_0 n_0}. \quad (20)
\end{aligned}$$

One has obviously $n_0 > 0$, $m_0 > 0$, and, for any $h \in \mathcal{G}$,

$$\|E_T(m(\Delta_j)) M(\lambda_j) h\|_{\mathcal{G}} = \left\| \frac{m(\lambda_j) - T}{n(\lambda_j)} E_T(m(\Delta_j)) h \right\|_{\mathcal{G}} \leq \frac{m_1 |\Delta_j| \cdot \|E_T(m(\Delta_j)) h\|_{\mathcal{G}}}{n_0}.$$

Using this estimate we obtain

$$\begin{aligned}
&\left| \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)(\bar{z}_1 - \lambda_j)(z_2 - \lambda_j)} \cdot \left\langle M(z_1) \xi_1, E_T(m(\Delta_j)) M(\lambda_j) \xi_2 \right\rangle_{\mathcal{G}} \right| \\
&\leq A \lim_{\delta \rightarrow 0} \sum_{j=1}^n |\Delta_j| \cdot \|E_T(m(\Delta_j)) M(z_1) \xi_1\|_{\mathcal{G}} \cdot \|E_T(m(\Delta_j)) \xi_2\|_{\mathcal{G}} \\
&\leq A \lim_{\delta \rightarrow 0} \left(\delta \sqrt{\sum_{j=1}^n \|E_T(m(\Delta_j)) M(z_1) \xi_1\|_{\mathcal{G}}^2} \cdot \sqrt{\sum_{j=1}^n \|E_T(m(\Delta_j)) \xi_2\|_{\mathcal{G}}^2} \right) \\
&= A \lim_{\delta \rightarrow 0} \left(\delta \|E_T(m([a, b])) M(z_1) \xi_1\|_{\mathcal{G}} \cdot \|E_T(m([a, b])) \xi_2\|_{\mathcal{G}} \right) \\
&= 0.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)(\bar{z}_1 - \lambda_j)(z_2 - \lambda_j)} \cdot \left\langle M(\lambda_j) \xi_1, E_T(m(\Delta_j)) M(z_2) \xi_2 \right\rangle_{\mathcal{G}} \\
&= \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)(\bar{z}_1 - \lambda_j)(z_2 - \lambda_j)} \cdot \left\langle M(\lambda_j) \xi_1, E_T(m(\Delta_j)) M(\lambda_j) \xi_2 \right\rangle_{\mathcal{G}} = 0.
\end{aligned}$$

Injecting these estimates into (19) one arrives at

$$\begin{aligned}\langle f, \Pi g \rangle_{\mathcal{H}} &= \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)(\bar{z}_1 - \lambda_j)(z_2 - \lambda_j)} \cdot \left\langle M(z_1) \xi_1, E_T(m(\Delta_j)) M(z_2) \xi_2 \right\rangle_{\mathcal{G}} \\ &= \left\langle \frac{m(z_1) - T}{n(z_1)} \xi_1, \int_a^b \frac{n(\lambda)}{m'(\lambda)(\bar{z}_1 - \lambda)(z_2 - \lambda)} dE_T(m(\lambda)) \frac{m(z_2) - T}{n(z_2)} \xi_2 \right\rangle_{\mathcal{G}}.\end{aligned}$$

Comparing this expression with (18) we conclude that P and Π coincide in \mathcal{D} . Therefore, the assertion of Lemma 4 holds under the additional assumption (11).

Now let $[a, b] \subset J$ be an arbitrary interval satisfying (8). By Lemma 2, one can find an open interval $(c, d) \subset [a, b]$ with the following properties:

- $c, d \notin \text{spec}_p H$,
- $m^{-1}(S_T) \subset J \subset (c, d)$,
- $m'(x) \neq 0$ for all $x \in [c, d]$.

By the first part of the proof, we have $E_H([c, d]) = \Pi([c, d]) := \Phi([c, d])\Phi([c, d])^*$. On the other hand, the interval $[c, d]$ contains all $\lambda \in [a, b]$ with $m(\lambda) \in \text{spec } T$, or, equivalently, all $\lambda \in [a, b] \cap \text{spec } H$. This means that $E_T([a, c]) = E_T([d, b]) = \Pi([a, c]) = \Pi([d, b]) = 0$, and, finally,

$$\begin{aligned}E_H([a, b]) &= E_H([a, c]) + E_H([c, d]) + E_H([d, b]) = E_H([c, d]) \\ &= \Pi([c, d]) = \Pi([a, c]) + \Pi([c, d]) + \Pi([d, b]) = \Pi([a, b]).\end{aligned}$$

This concludes the proof of Lemma 4. \square

3 Unitary equivalence

To avoid potential confusions let us introduce a definition.

Definition 5. Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces. Let $\mathcal{L}_j \subset \mathcal{H}_j$ be closed linear subspaces viewed as Hilbert spaces with the induced scalar products, and let $P_j : \mathcal{H}_j \rightarrow \mathcal{L}_j$ be the orthogonal projections, $j = 1, 2$. We say that an operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ defines a unitary map from \mathcal{L}_1 to \mathcal{L}_2 if

- $U(\mathcal{L}_1) \subset \mathcal{L}_2$ and
- the map $P_2 U P_1^* : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a unitary operator.

Lemma 6. Let \mathcal{H}_j , \mathcal{L}_j , P_j , $j = 1, 2$, be as in Definition 5, and let an operator $U \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ satisfy

$$(i) \ U^* U = P_1^* P_1 \text{ and } (ii) \ U U^* = P_2^* P_2, \quad (21)$$

then

- (a) U defines a unitary map from \mathcal{L}_1 to \mathcal{L}_2 and

$$(b) \quad UP_1^*P_1 = P_2^*P_2U.$$

Proof. This is an elementary result and we give the proof just for the sake of completeness. The part (b) follows directly from (21), so we just need to prove the part (a). Let us show first the inclusion $U(\mathcal{L}_1) \subset \mathcal{L}_2$. Let $x \in \mathcal{L}_1$ such that $Ux \in \mathcal{L}_2^\perp$. In view of the definition of P_2 the last equality means that $P_2Ux = 0$. On the other hand, with the help of (21) we obtain:

$$\begin{aligned} 0 &= \|P_2Ux\|_{\mathcal{L}_2}^2 = \langle P_2Ux, P_2Ux \rangle_{\mathcal{L}_2} \\ &= \langle x, U^*(P_2^*P_2)Ux \rangle_{\mathcal{H}_2} = \langle x, U(U^*U)Ux \rangle_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}^2, \end{aligned}$$

which shows the equality $U(\mathcal{L}_1) \cap \mathcal{L}_2^\perp = \{0\}$ and the sought inclusion.

Now we have, with the help of (21),

$$\begin{aligned} (P_2UP_1^*)(P_2UP_1^*)^* &= P_2U(P_1^*P_1)U^*P_2^* \\ &= P_2U(U^*U)U^*P_2^* = P_2P_2^* \equiv \text{Id}_{\mathcal{L}_2}, \end{aligned}$$

and

$$\begin{aligned} (P_2UP_1^*)^*(P_2UP_1^*) &= P_1U^*(P_2^*P_2)UP_1^* \\ &= P_1U^*(UU^*)UP_1^* = P_1P_1^* \equiv \text{Id}_{\mathcal{L}_1}, \end{aligned}$$

which shows the unitarity of $P_2UP_1^*$. □

Lemma 7. *If a segment $[a, b] \subset J$ satisfies (8), then:*

- (a) $\Phi([a, b])^*\Phi([a, b]) = E_T(m([a, b]))$
- (b) $\Phi([a, b])$ defines a unitary map from $\text{ran } E_T(m([a, b])) \subset \mathcal{G}$ and $\text{ran } E_H([a, b]) \subset \mathcal{H}$,
- (c) $E_H([a, b]) = \Phi([a, b])E_T(m([a, b]))\Phi([a, b])^*$.

Proof. Taking into account Lemmas 4 and 6 we see that the assertions (b) and (c) follow if we prove (a).

Take again a partition Δ of $[a, b]$, $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$, denote $\Delta_j := [\lambda_{j-1}, \lambda_j]$ and consider the Riemann-Stieltjes integral sum (10) with $\xi_j := \lambda_j$,

$$\Phi_\Delta = \sum_{j=1}^n \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \gamma(\lambda_j) E_T(m(\Delta_j)).$$

As noted above, the sums Φ_Δ converge in the operator norm to $\Phi := \Phi([a, b])$, the adjoint operators Φ_Δ^* converge then to Φ^* , and the products $\Phi_\Delta^*\Phi_\Delta$ converge to $\Phi^*\Phi$ as $\delta := \max_j |\Delta_j|$ tends to 0. We have

$$\Phi_\Delta^*\Phi_\Delta = \sum_{j,k=1}^n \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \sqrt{\frac{n(\lambda_k)}{m'(\lambda_k)}} E_T(m(\Delta_j)) \gamma(\lambda_j)^* \gamma(\lambda_k) E_T(m(\Delta_k)).$$

Using (7) we can write

$$\gamma(\lambda_j)^* \gamma(\lambda_k) = L(\lambda_j, \lambda_k) := \begin{cases} \frac{M(\lambda_k) - M(\lambda_j)}{\lambda_k - \lambda_j}, & j \neq k, \\ M'(\lambda_j) \equiv \frac{m'(\lambda_j)}{n(\lambda_j)} - \frac{n'(\lambda_j)}{n(\lambda_j)^2} (m(\lambda_j) - T), & j = k. \end{cases}$$

Note that the operators $L(\lambda_j, \lambda_k)$ commute with T for any $j, k = 1, \dots, n$ and that we have the equality $E_T(m(\Delta_j))E_T(m(\Delta_k)) = 0$ for all $j \neq k$. This simplifies the above expression:

$$\begin{aligned} \Phi_\Delta^* \Phi_\Delta &= \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} M'(\lambda_j) E(m(\Delta_j)) \\ &= \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \frac{m'(\lambda_j)}{n(\lambda_j)} E(m(\Delta_j)) - \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \frac{n'(\lambda_j)}{n(\lambda_j)^2} (m(\lambda_j) - T) E(m(\Delta_j)) \\ &= \sum_{j=1}^n E(m(\Delta_j)) - \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \frac{n'(\lambda_j)}{n(\lambda_j)^2} (m(\lambda_j) - T) E(m(\Delta_j)) \\ &= E\left(m([a, b])\right) - \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \frac{n'(\lambda_j)}{n(\lambda_j)^2} (m(\lambda_j) - T) E(m(\Delta_j)). \end{aligned}$$

Using the constants (20) and $n_2 := \max_{x \in [a, b]} |n'(x)|$ one can estimate, for any $h \in \mathcal{G}$,

$$\left\| (m(\lambda_j) - T) E(m(\Delta_j)) h \right\|_{\mathcal{G}} \leq m_1 |\Delta_j| \cdot \left\| E(m(\Delta_j)) h \right\|_{\mathcal{G}}$$

and, using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \left\| \sum_{j=1}^n \frac{n(\lambda_j)}{m'(\lambda_j)} \frac{n'(\lambda_j)}{n(\lambda_j)^2} (m(\lambda_j) - T) E(m(\Delta_j)) h \right\|_{\mathcal{G}} &\leq \frac{n_2 m_1}{n_0 m_0} \sum_{j=1}^n |\Delta_j| \left\| E(m(\Delta_j)) h \right\|_{\mathcal{G}} \\ &\leq \frac{n_2 m_1}{n_0 m_0} \sqrt{\sum_{j=1}^n |\Delta_j|} \cdot \sqrt{\sum_{j=1}^n |\Delta_j| \left\| E(m(\Delta_j)) h \right\|_{\mathcal{G}}^2} \\ &\leq \frac{n_2 m_1 \sqrt{b-a}}{n_0 m_0} \max_j |\Delta_j| \sqrt{\sum_{j=1}^n \left\| E(m(\Delta_j)) h \right\|_{\mathcal{G}}^2} \\ &\leq \frac{n_2 m_1 \sqrt{b-a}}{n_0 m_0} \max_j |\Delta_j| \left\| E(m([a, b])) h \right\|_{\mathcal{G}}. \end{aligned}$$

Hence, $\Phi \Phi^* = \lim_{\max |\Delta_j| \rightarrow 0} \Phi_\Delta \Phi_\Delta^* = E(m([a, b]))$, which proves (a). \square

Lemma 8. *Let intervals $[a, b] \subset J$ and $[c, d] \subset J$ be such that*

- a, b, c, d do not belong to $\text{spec}_p H$,
- $(a, b) \cap (c, d) = \emptyset$,

then $\Phi([a, b])E_T(m([c, d])) = 0$, $\Phi([a, b])\Phi([c, d])^ = 0$ and $\Phi([a, b])^*\Phi([c, d]) = 0$.*

Proof. Take a partition Δ of $[a, b]$, $a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b$, denote $\Delta_j := [\lambda_{j-1}, \lambda_j]$ and consider the Riemann-Stieltjes integral sum (10) with $\xi_j := \lambda_j$,

$$\Phi_\Delta = \sum_{j=1}^n \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \gamma(\lambda_j) E_T(m(\Delta_j)).$$

Futhermore, take a partition Π of $[c, d]$, $c = \mu_0 < \mu_1 < \dots < \mu_m = d$, denote $\Pi_j := [\mu_{j-1}, \mu_j]$ and consider the Riemann-Stieltjes integral sum (10) with $\xi_j := \mu_j$,

$$\Phi_\Pi = \sum_{j=1}^m \sqrt{\frac{n(\mu_j)}{m'(\mu_j)}} \gamma(\mu_j) E_T(m(\Pi_j)).$$

We know that Φ_Δ and Φ_Π converge in the norm to $\Phi([a, b])$ and $\Phi([c, d])$ respectively as both $\delta := \max_j |\Delta_j|$ and $\varepsilon := \max_j |\Pi_j|$ tend to 0.

On the other hand, under the assumptions made we have $E_T(m(\Delta_j)) E_T(m([c, d])) = 0$ for any j , which implies $\Phi_\Delta E_T(m([c, d])) = 0$ and $\Phi([a, b]) E_T(m([c, d])) = 0$.

In a similar way, using the equalities $E_T(m(\Delta_j)) E_T(m(\Pi_k)) = 0$ for all $j = 1, \dots, n$ and $k = 1, \dots, m$, we obtain $\Phi_\Delta \Phi_\Pi^* = 0$, which implies $\Phi([a, b]) \Phi([c, d])^* = 0$.

Finally, we have, using (7),

$$\begin{aligned} \Phi_\Delta^* \Phi_\Pi &= \sum_{j=1}^n \sum_{k=1}^m \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \cdot \sqrt{\frac{n(\mu_k)}{m'(\mu_k)}} E_T(m(\Delta_j)) \gamma(\lambda_j)^* \gamma(\mu_k) E_T(m(\Pi_k)) \\ &= \sum_{j=1}^n \sum_{k=1}^m \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \cdot \sqrt{\frac{n(\mu_k)}{m'(\mu_k)}} E_T(m(\Delta_j)) \frac{M(\mu_k) - M(\lambda_j)}{\mu_k - \lambda_j} E_T(m(\Pi_k)) \\ &= \sum_{j=1}^n \sum_{k=1}^m \sqrt{\frac{n(\lambda_j)}{m'(\lambda_j)}} \cdot \sqrt{\frac{n(\mu_k)}{m'(\mu_k)}} \frac{M(\mu_k) - M(\lambda_j)}{\mu_k - \lambda_j} E_T(m(\Delta_j)) E_T(m(\Pi_k)) = 0, \end{aligned}$$

which proves the remaining equality $\Phi([a, b])^* \Phi([c, d]) = 0$. \square

All the intervals we considered so far were contained in the gap J together with their closures. Let us extend the above considerations to arbitrary subintervals of J .

Lemma 9. *The strong limit*

$$\Phi(J) := s\text{-}\lim_{\varepsilon \rightarrow +0} \Phi([a_0 + \varepsilon, b_0 - \varepsilon])$$

exists and defines a unitary map from $\text{ran } E_T(m(J))$ to $\text{ran } E_H(J)$

Proof. Let us take a monotonically decreasing sequence of $(\varepsilon_n)_{n \geq 0}$ converging to 0 such that $a_0 + \varepsilon_n, b_0 - \varepsilon_n \notin \text{spec}_p H$. Denote $J_n := [a_0 + \varepsilon_n, b_0 - \varepsilon_n]$ and

$$\Phi(J_n \setminus J_{n-1}) := \Phi([a_0 + \varepsilon_n, a_0 + \varepsilon_{n-1}]) + \Phi([b_0 - \varepsilon_{n-1}, b_0 - \varepsilon_n]).$$

Let $\xi \in \mathcal{G}$, then

$$\Phi(J_n) \xi = \Phi(J_0) \xi + \sum_{j=0}^n \Phi(J_n \setminus J_{n-1}) \xi. \quad (22)$$

By Lemma 8, the summands on the right-hand side of (22) are paarwise orthogonal. Moreover, by Lemma 7(a) we have

$$\left\| \Phi(J_0)\xi \right\|_{\mathcal{H}}^2 = \left\| E_T(m(J_0))\xi \right\|_{\mathcal{G}}^2 \text{ and } \left\| \Phi(J_n \setminus J_{n-1})\xi \right\|_{\mathcal{H}}^2 = \left\| E_T(m(J_n \setminus J_{n-1}))\xi \right\|_{\mathcal{G}}^2,$$

and once can estimate

$$\begin{aligned} \left\| \Phi(J_0)\xi \right\|_{\mathcal{H}}^2 + \sum_{j=0}^n \left\| \Phi(J_n \setminus J_{n-1})\xi \right\|_{\mathcal{H}}^2 &= \left\| E_T(m(J_0))\xi \right\|_{\mathcal{G}}^2 + \sum_{j=0}^n \left\| E_T(m(J_n \setminus J_{n-1}))\xi \right\|_{\mathcal{G}}^2 \\ &= \left\| E_T(m(J_n))\xi \right\|_{\mathcal{G}}^2 \leq \left\| E_T(m(J))\xi \right\|_{\mathcal{G}}^2 < \infty. \end{aligned}$$

Hence, the series on the right-hand side of (22) converges, and one shows that the limit is in fact independent of the choice of the sequence ε_n in the standard way using the relations

$$\text{s-lim}_{\varepsilon \rightarrow +0} E_T((a_0, a_0 + \varepsilon)) = \text{s-lim}_{\varepsilon \rightarrow +0} E_T((b_0 - \varepsilon, b_0)) = 0.$$

Therefore, the map $\Phi(J)$ is well defined. Let us show that the assumptions of Lemma 6 are satisfied. First, proceeding as above we can show the equality

$$\Phi(J)^* := \text{s-lim}_{\varepsilon \rightarrow +0} \Phi([a_0 + \varepsilon, b_0 - \varepsilon])^*.$$

Second, by combining the results of Lemma 4, Lemma 7(a) and Lemma 8 we see that for $n < m$ one has the identities $\Phi(J_n)^*\Phi(J_m) = E_T(m(J_n))$ and $\Phi(J_n)\Phi(J_m)^* = E_H(J_n)$. Taking first the strong limit for $m \rightarrow +\infty$ and then the strong limit as $n \rightarrow +\infty$, we arrive at $\Phi(J)^*\Phi(J) = E_T(m(J))$ and $\Phi(J)\Phi(J)^* = E_H(J)$, which gives the conclusion by applying Lemma 6. \square

The following theorem summarizes the above constructions.

Theorem 10. *For any borelian subset Ω of J one has $E_H(\Omega) = \Phi(J)E_T(m(\Omega))\Phi(J)^*$.*

Proof. Assume first that $\Omega \subset J$ is a closed interval whose endpoints are not eigenvalues of H . Take the same family of intervals (J_n) as in the above proof of Lemma 9, then for sufficiently large m and n one has $\Omega \subset J_m$ and $\Omega \subset J_n$. By Lemma 7 and Lemma 8 we have $E_H(\Omega) = \Phi(J_m)E_T(m(\Omega))\Phi(J_n)^*$. Taking the repeated strong limit, first for $n \rightarrow \infty$ and then for $m \rightarrow \infty$, one arrives at $E_H(\Omega) = \Phi(J)E_T(m(\Omega))\Phi(J)^*$. This identity is then extended to all borelian subsets $\Omega \subset J$ using the σ -additivity. \square

As an immediate corollary we have the following relation between H and T :

Corollary 11. *Introduce the orthogonal projections $P_H : \mathcal{H} \rightarrow \text{ran } E_H(J)$ and $P_T : \mathcal{G} \rightarrow \text{ran } E_T(m(J))$, then the operators $m(H_J)$ and $T_{m(J)}$ are related by $m(H_J) = UT_{m(J)}U^*$, where U is the unitary operator from $\text{ran } E_T(m(J))$ to $\text{ran } E_H(J)$ defined by $U := P_H\Phi(J)P_T^*$.*

Proof. The unitarity of the map U was already shown in Lemma 9, and the requested equality is obtained from the spectral theorem and Theorem 10 as follows

$$\begin{aligned}
m(H_J) &= P_H \int_J m(\lambda) dE_H(\lambda) P_H^* \\
&= P_H \Phi(J) \int_J m(\lambda) dE_T(m(\lambda)) \Phi(J)^* P_H^* = P_H \Phi(J) \int_{m(J)} \lambda dE_T(\lambda) \Phi(J)^* P_H^* \\
&= P_H \Phi(J) P_T^* \int_{\mathbb{R}} \lambda dE_{T_{m(J)}}(\lambda) P_T \Phi(J)^* P_H^* = U T_{m(J)} U^*. \quad \square
\end{aligned}$$

Remark 12. The assertions of Corollary 11 can also be rewritten in an equivalent form as $H_J = U m^{-1}(T_{m(J)}) U^*$. Taking into account the identity

$$m^{-1}(T_{m(J)}) = \int_J \lambda dE(m(\lambda))$$

and applying the elementary arguments with the Riemann-Stieltjes integral sums one can represent H_J as a two-side operator Riemann-Stieltjes integral

$$H_J = P_H \int_J \frac{\lambda n(\lambda)}{m'(\lambda)} \gamma(\lambda) dE_T(m(\lambda)) \gamma(\lambda)^* P_H^*,$$

which may be viewed as a direct generalization of Proposition 3 to the case of arbitrary spectra.

4 Applications

4.1 Direct sums and arrays of quantum dots

Let L be a closed symmetric densely defined operator in a Hilbert space \mathcal{K} with deficiency indices $(1, 1)$, and let (\mathbb{C}, π, π') be a boundary triplet for L ; we denote by ν and m be the associated γ -field and Weyl function, respectively. Denote by K^0 the self-adjoint extension of L defined by the boundary conditions $\pi f = 0$.

Now let \mathcal{A} be a non-empty countable set. Introduce a Hilbert space \mathcal{H} and an operator S in \mathcal{H} by

$$\mathcal{H} := \bigoplus_{\alpha \in \mathcal{A}} \mathcal{H}_\alpha, \quad S = \bigoplus_{\alpha \in \mathcal{A}} S_\alpha,$$

where each \mathcal{H}_α is a copy of \mathcal{K} and $S_\alpha := L$. Elementary considerations show that S is a closed densely defined symmetric operator whose deficiency indices are (a, a) , where a is the cardinality of the set \mathcal{A} , and as a boundary triplet for S one can take $(\mathcal{G}, \Gamma, \Gamma')$ with

$$\mathcal{G} := \ell^2(\mathcal{A}), \quad \Gamma(f_\alpha) = (\pi f_\alpha), \quad \Gamma'(f_\alpha) = (\pi' f_\alpha).$$

Clearly, the associated γ -field $\gamma(z)$ is just the direct sum, $\gamma(z)(\xi_\alpha) = (\nu(z)\xi_\alpha)$, and the Weyl function is $M(z) = m(z) \text{Id}$, where Id the identity operator in \mathcal{G} . We refer to [29, Section 3] for a detailed discussion of boundary triplet machinery for direct sums. We also

remark that the above construction exhausts, up to a unitary equivalence, all the cases in which the Weyl function is of scalar type, i.e. is just the multiplication by a scalar function, see [2].

Our aim now is to study the self-adjoint extensions of S . Note first that, due to the above construction, the distinguished extension of S defined by $H^0 := S^* \upharpoonright \ker \Gamma$ is just the direct sum of the copies of the operator K^0 and, in particular, one has the equality $\text{spec } H^0 = \text{spec } K^0$. Now let T be a bounded self-adjoint operator in \mathcal{G} . Consider the self-adjoint operator

$$H_T := S^* \upharpoonright \ker(\Gamma' - T\Gamma);$$

it is known that H_T is a self-adjoint extension of S [17], and we are going to show that its spectral analysis is covered by the scheme of the present paper. Define

$$\Gamma_T := \Gamma, \quad \Gamma'_T := \Gamma' - T\Gamma,$$

then one can easily see that $(\mathcal{G}, \Gamma_T, \Gamma'_T)$ is another boundary triplet for S , that the associated γ -field is the same as for $(\mathcal{G}, \Gamma, \Gamma')$, and that the associated Weyl function M_T has the form $M_T(z) := m(z) - T$. At the same time, with respect to this new boundary triplet the operator H^0 corresponds to the boundary conditions $\Gamma_T f = 0$, and H_T corresponds to the boundary conditions $\Gamma'_T f = 0$.

The physical interpretation of the above situation is as follows. The operator K^0 may be viewed as a Hamiltonian of a single quantum dot, and a copy of the quantum dot is placed at each node of a discrete structure \mathcal{A} . Then the operator H^0 describes the array of non-interacting quantum dots, while H_T is Hamiltonian of the quantum dots interacting with each other through the boundary conditions $\Gamma'_T f = T\Gamma f$, and one can be interested in the dependence of the spectral properties of H_T on the “interaction parameter” T describing the inter-node interactions. Such an approach is known under the name “restriction-extension procedure” which proved its usefulness in the study of solvable models in quantum mechanics [4, 16, 38]. So Corollary 11 being applied to the present situation gives the following results:

Theorem 13. *For any bounded self-adjoint operator T in $\ell^2(\mathcal{A})$ and any interval $J \subset \mathbb{R} \setminus \text{spec } K^0$ one has $(H_T)_J = U m^{-1}(T_{m(J)}) U^*$, where U acts as*

$$U = \int_J \sqrt{\frac{1}{m'(\lambda)}} \gamma(\lambda) dE_T(m(\lambda))$$

being considered as a unitary map from $\text{ran } E_T(m(J))$ to $\text{ran } E_{H_T}(J)$.

For example, the papers [22, 23] study the case when K^0 is a magnetic harmonic oscillator,

$$\mathcal{K} = L^2(\mathbb{R}^2), \quad K^0 = -\left(\frac{\partial}{\partial x_1} - iBx_2\right)^2 - \frac{\partial^2}{\partial x_2^2} + \omega^2(x_1^2 + x_2^2), \quad \omega, B > 0,$$

and L is the restriction of K^0 on the functions vanishing at the origin. Under a certain standard choice of the boundary triplet, the associated maps $\nu(z)$ and $m(z)$ are of the form

$$\nu(z)\xi = \xi G(\cdot, 0; z), \quad m(z) = \lim_{x \rightarrow 0} \left(G(x, 0; z) + \frac{\log |x|}{2\pi} \right)$$

where G is the Green function of K^0 , i.e. the integral kernel of the resolvent $(K^0 - z)^{-1}$ for $z \notin \text{spec } K^0$; explicit analytic expressions for G and m are given in [22]. The operator K^0 has a discrete spectrum, which means that $\text{spec } H^0$ is a discrete set. Outside this “forbidden set” the Hamiltonian H_T is completely described in terms of T .

4.2 Differential operators on networks

Another series of examples comes from the study of differential operators on networks, also called metric graphs or quantum graphs, see the recent monograph [8] and the collections [20, 40]. Let us describe a representative particular case where our machinery works; we use the notation proposed in [14].

Let X be a countable connected graph with symmetric neighborhood relation \sim and without loops and multiple edges. We shall view it as a one-complex, where each edge is a homeomorphic copy of the unit interval, and the edges are glued together at common endpoints. We write X^0 for the vertex set and X^1 for the one-skeleton of X . Every point of X^1 is of the form (xy, t) , the point at distance t from x on the non-oriented edge $[x, y] = [y, x]$, where $t \in [0, 1]$ and $x, y \in X^0$ with $x \sim y$. Thus one has $(xy, 0) = x$ and $(xy, t) = (yx, 1 - t)$. In this way, the discrete graph metric on the vertex set (minimal length = number of edges of a connecting path) has a natural extension to X^1 .

On X^0 one has a natural discrete measure m^0 defined by

$$m^0(x) = \#\{y \in X^0 : y \sim x\}$$

and we assume that

$$m^0(x) < \infty \text{ for any } x \in X^0.$$

On X^1 we introduce the continuous Lebesgue measure m^1 which at the point (xy, t) is given by dt if $t \in (0, 1)$, and the vertex set has m^1 -measure equal to zero. The graph X equipped with the above constructions will be called a network or a metric graph.

For any function $F : X^1 \rightarrow \mathbb{C}$ and for $x \in X^0$ and $y \sim x$ denote by F_{xy} the function $t \mapsto F(xy, t)$, then the Hilbert space $L^2(X^1, m^1)$ is exactly the space of the measurable functions F such that

$$\|F\|_{L^2(X^1, m^1)}^2 := \sum_{x \sim y} \|F_{xy}\|_{L^2(0,1)}^2 < \infty;$$

we assume that each edge appears only once in the sum. Associated with a network, there are at least two natural operators. The first one is the discrete transition operator P acting on functions $g : X^0 \rightarrow \mathbb{C}$ by

$$Pg(x) = \frac{1}{m^0(x)} \sum_{y: y \sim x} g(y).$$

The above expression defines a bounded self-adjoint operator in $\ell^2(X^0, m^0)$ with the norm ≤ 1 , to be denoted by the same symbol P . The second operator is the continuous (positive) Laplace operator L acting in the space $L^2(X^1, m^1)$ as the second derivative. More precisely, for any function F the prime sign will denote the derivation with respect to the length parameter, i.e.

$$F'(xy, t) := F'_{xy}(t).$$

We introduce the space

$$\widehat{H}^k(X^1, m^1) := \left\{ F \in \mathcal{L}^2(X^1, m^1) : F^{(k)} \in \mathcal{L}^2(X^1, m^1) \right\},$$

then the operator L acts as $LF = -F''$ on functions $F \in \widehat{H}^2(X^1, m^1)$ satisfying the boundary conditions

$$\begin{aligned} F(xu, 0) = F(xv, 0) &=: F(x) \text{ for all } x, u, v \in X^0 \text{ with } u \sim x \text{ and } v \sim x, \\ F'(x) = 0 \text{ for all } x \in X^0, \quad \text{where } F'(x) &:= \sum_{y: y \sim x} F'(xy, 0+). \end{aligned} \quad (23)$$

It has been known for a long time that the discrete operator P and the network laplacian L are closely related [6, 15, 19, 35]. In particular, for the case of a finite X it was shown in [6] that

$$\text{spec } L \setminus \Sigma = \{z \notin \Sigma : \cos \sqrt{z} \in \text{spec } P\} \quad \text{with} \quad \Sigma := \{(\pi n)^2 : n \in \mathbb{N}\},$$

and it was extended in [15] to the case of arbitrary graphs. The set Σ plays a special role, and its relation with the spectrum of L is also known but the presentation of the respective results would need some special vocabulary from the graph theory; an interested reader may consult the respective section in [15]. It was pointed out in the author's paper [36] that the boundary triplet machinery can be applied to the study of the operator L . This approach was used to improve the above relation: it was first shown that

$$\text{spec}_\star L \setminus \Sigma = \{z \notin \Sigma : \cos \sqrt{z} \in \text{spec}_\star P\}, \quad \star \in \{\text{p, pp, disc, ess, ac, sc}\},$$

see [13, Section 3.5], and later it was shown that for any interval $J \subset \mathbb{R} \setminus \Sigma$ the operator $\cos \sqrt{L_J}$ is unitarily equivalent to $P_{\cos \sqrt{J}}$, see [37, Theorem 17]; here and below we use the notation

$$\cos \sqrt{J} := \{\cos \sqrt{\lambda} : \lambda \in J\}.$$

Let us recall how the boundary triplet machinery applies to this case and explain what kind of improvements can be obtained.

Denote by S the restriction of L to the functions F for which $F(x) = 0$ for all $x \in X^0$. Clearly, this is a closed densely defined symmetric operator. The following assertions are proved in [36]:

- the adjoint operator S^* acts as $F \mapsto -F''$ on the domain

$$\text{dom } S^* = \left\{ F \in \widehat{H}^2(X^1, m^1) : \text{Eq. (23) holds} \right\},$$

- as a boundary triplet for S one can take $(\mathcal{G}, \Gamma, \Gamma')$ with

$$\mathcal{G} = \ell^2(X^0, m^0), \quad \Gamma f := (F(x)), \quad \Gamma' f := \left(\frac{F'(x)}{m^0(x)} \right),$$

- the associated γ -field is given by

$$[\gamma(z)\xi](xy, t) = \xi(x) \frac{\sin(\sqrt{z}(1-t))}{\sin \sqrt{z}} + \xi(y) \frac{\sin(\sqrt{z}t)}{\sin \sqrt{z}},$$

- the associated Weyl function takes the form

$$M(z) = -\frac{\sqrt{z}}{\sin \sqrt{z}} (\cos \sqrt{z} - P).$$

It is easy to see that:

- L is exactly the restriction of S^* to $\ker \Gamma'$,
- the restriction $L^0 := S^* \upharpoonright \ker \Gamma$ is the direct sum (over all edges xy) of the Dirichlet Laplacians on $(0, 1)$, and $\text{spec } L^0 = \Sigma$.

So we are in the situation covered by the present paper. Using the equality $n = 2m'$ and applying Corollary 11 we arrive at the following result:

Theorem 14. *For any interval $J \subset \mathbb{R} \setminus \Sigma$ one has $\cos \sqrt{L_J} = UP_{\cos \sqrt{J}}U^*$, where the operator U acts by*

$$U = \sqrt{2} \int_J \gamma(\lambda) dE_P(\cos \sqrt{\lambda}).$$

and defines a unitary map from $\text{ran } E_P(\cos \sqrt{J})$ to $\text{ran } L_J$.

We note that the above operator L is just a particular case of a differential operator acting on a network. It was shown in several papers that Weyl functions of the form (4) appear also in other cases, for example, for operators with magnetic fields, more general boundary conditions, additional scalar potentials, we refer e.g. to [12, 36, 37, 39] and to the references there-in. In all those cases one obtains a similar result on a unitary equivalence between a differential operator and a certain discrete operator: the operator P becomes then a generalized discrete Laplacian [39], and the function $z \mapsto \cos \sqrt{z}$ should be replaced by a certain analytic function expressed in terms of fundamental solutions [37, Section 3.2].

Remark 15. The above operator P is just one operator from a large family of Laplace-type operators associated with a graph. There exists various unbounded counterparts for which the problem of self-adjoint extensions has its own interest, cf. the recent works [24, 28].

Remark 16. The operators L of the above type could be viewed as operator-valued ordinary differential operators, whose study is rather active during the last years [7, 21, 26, 32, 34], but are not aware of any previous result giving a unitary equivalence between such an operator and the coefficients in the respective boundary conditions.

Acknowledgments

The work was partially supported by ANR NOSEVOL 2011 BS01019 01 and GDR Dynamique quantique.

References

- [1] V. Adamyan, H. Langer, R. Mennicken, J. Saurer: *Spectral components of self-adjoint block operator matrices with unbounded entries*. Math. Nachr. **178** (1996) 43–80.

- [2] S. Albeverio, J. F. Brasche, M. M. Malamud, H. Neidhardt: *Inverse spectral theory for asymmetric operators with several gaps: scalar-type Weyl functions*. J. Funct. Anal. **228** (2005) 144–188.
- [3] S. Albeverio, J. F. Brasche, H. Neidhardt: *On inverse spectral theory for self-adjoint extensions*. J. Funct. Anal. **154** (1998) 130–173.
- [4] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable models in quantum mechanics*. With an appendix by P. Exner (AMS, Providence, 2005).
- [5] S. Albeverio, A. K. Motovilov: *Operator Stieltjes integrals with respect to a spectral measure and solutions of some operator equations*. Trans. Moscow Math. Soc. **2011** (2011) 45–77.
- [6] J. von Below: *A characteristic equation associated to an eigenvalue problem on c^2 -networks*. Lin. Alg. Appl. **71** (1985) 309–325.
- [7] J. von Below, D. Mugnolo: *The spectrum of the Hilbert space valued second derivative with general self-adjoint boundary conditions*. Preprint arXiv:1209.0932.
- [8] G. Berkolaiko, P. Kuchment: *Introduction to quantum graphs* (Volume 186 of Mathematical Surveys and Monographs, AMS, 2013).
- [9] M. Sh. Birman, M. Solomyak: *Double operator integrals in a Hilbert space*. Integral Equations Operator Theory **47**:2 (2003) 131–168.
- [10] J. F. Brasche: *Spectral theory for self-adjoint extensions*. In the book R. del Rio, C. Villegas (eds.): *Spectral theory of Schrödinger operators* (Volume 340 of Contemporary Mathematics, AMS, 2004), pages 51–96.
- [11] J. F. Brasche, M. Malamud, H. Neidhardt: *Weyl functions and spectral properties of self-adjoint extensions*. Integral Equations Operator Theory **43** (2002) 264–289.
- [12] J. Brüning, V. Geyler, K. Pankrashkin: *Cantor and band spectra for periodic quantum graphs with magnetic fields*. Commun. Math. Phys. **269** (2007) 87–105.
- [13] J. Brüning, V. Geyler, K. Pankrashkin: *Spectra of self-adjoint extensions and applications to solvable Schrödinger operators*. Rev. Math. Phys. **20** (2008) 1–70.
- [14] D. I. Cartwright, W. Woess: *The spectrum of the averaging operator on a network (metric graph)*. Illinois J. Math. **51** (2007) 805–830.
- [15] C. Cattaneo: *The spectrum of the continuous Laplacian on a graph*. Monatsh. Math. **124** (1997) 215–235.
- [16] G. Dell’Antonio, P. Exner, V. Geyler (Eds): *Special issue on singular interactions in quantum mechanics: solvable models*. J. Phys. A **38** (2005), issue no. 22.
- [17] V. A. Derkach, M. M. Malamud: *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*. J. Funct. Anal. **95** (1991) 1–95.
- [18] V. A. Derkach, M. M. Malamud: *The extension theory of Hermitian operators and the moment problem*. J. Math. Sci. **73**:2 (1995) 141–242.
- [19] P. Exner: *A duality between Schrödinger operators on graphs and certain Jacobi matrices*. Ann. Inst. Henri Poincaré, Section A Phys. Théor. **66** (1997) 359–371.
- [20] P. Exner, J. P. Keating, P. Kuchment, T. Sunada, A. Teplyaev (Eds.): *Analysis on graphs and its applications* (Volume 77 of Proceedings of Symposia in Pure Mathematics, AMS, 2008).
- [21] F. Gesztesy, R. Weikard, M. Zinchenko: *Initial value problems and Weyl-Titchmarsh theory for Schrödinger operators with operator-valued potentials*. Operators Matrices **7**:2 (2013) 241–283.
- [22] V. A. Geyler, I. Yu. Popov: *The spectrum of a magneto-Bloch electron in a periodic array of quantum dots: Explicitly solvable models*. Z. Phys. B: Cond. Mat. **93** (1994) 437–439.

- [23] V. A. Geyler, I. Yu. Popov: *Periodic array of quantum dots in a magnetic field: Irrational flux; honeycomb lattice*. Z. Phys. B: Cond. Mat. **98** (1995) 473–477.
- [24] S. Golénia, C. Schumacher: *The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs*. J. Math. Phys. **52**:6 (2011) 063512.
- [25] D. M. Gitman, I. V. Tyutin, B. L. Voronov: *Self-adjoint extensions in quantum mechanics: General theory and applications to Schrödinger and Dirac equations with singular potentials* (Volume 62 of Progress in Mathematical Physics, Birkhäuser, 2012).
- [26] V. I. Gorbachuk, M. A. Gorbachuk: *Boundary value problems for operator differential equations*. Kluwer Acad. Publ., Dordrecht etc., 1991.
- [27] V. Jakšić: *Topics in spectral theory*. In the book S. Attal, A. Joye, C.-A. Pillet (Eds.), *Open Quantum Systems I. Recent Developments* (Volume 1880 of Lecture Notes in Mathematics, Springer, Berlin, 2006), pages 235–312.
- [28] P. E. T. Jorgensen: *Unbounded graph-Laplacians in energy space, and their extensions*. J. Appl. Math. Comput. **39**:1–2 (2012) 155–187.
- [29] A. S. Kostenko, M. M. Malamud: *1-D Schrödinger operators with local point interactions on a discrete set*. J. Differential Equations **249** (2010) 253–304.
- [30] M. G. Krein: *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications*. Math. Sb. N. Ser. **20** (1947) 431–495.
- [31] M. M. Malamud, H. Neidhardt: *On the unitary equivalence of absolutely continuous parts of self-adjoint extensions*. J. Funct. Anal. **260** (2011) 613–638.
- [32] M. M. Malamud, H. Neidhardt: *Sturm-Liouville boundary value problems with operator potentials and unitary equivalence*. J. Differential Equations **252** (2012) 5875–5922.
- [33] M. M. Malamud, S. M. Malamud: *Spectral theory of operator measures in Hilbert space*. St. Petersburg Math. J. **15** (2004) 323–373.
- [34] V. Mogilevskii: *Boundary triplets and Titchmarsh-Weyl functions of differential operators with arbitrary deficiency indices*. Meth. Funct. Anal. Topol. **15** (2009) 280–300.
- [35] S. Nicaise: *Approche spectrale des problèmes de diffusion sur les réseaux*. In the book F. Hirsch, G. Mokobodzki (Eds.): *Séminaire de Théorie du Potentiel Paris, No. 8* (Volume 1235 of Lecture Notes in Mathematics, Springer, 1987), pages 120–140.
- [36] K. Pankrashkin: *Spectra of Schrödinger operators on equilateral quantum graphs*. Lett. Math. Phys. **77** (2006) 139–154.
- [37] K. Pankrashkin: *Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures*. J. Math. Anal. Appl. **396** (2012) 640–655.
- [38] B. S. Pavlov: *The theory of extensions and explicitly-soluble models*. Russ. Math. Surv. **42**:6 (1987) 127–168.
- [39] O. Post: *Equilateral quantum graphs and boundary triples*. In the book [20], pages 469–490.
- [40] O. Post: *Spectral analysis on graph-like spaces* (Volume 2039 of Lecture Notes in Mathematics, Springer, 2012).